BASIC PROPERTIES OF FEDOSOV SUPERMANIFOLDS[1]

B. Geyer [a)2] and P.M. Lavrov [a),b)3]

[a)] *Center of Theoretical Studies, Leipzig University, Augustusplatz 10/11, D-04109 Leipzig, Germany*

[b)] *Tomsk State Pedagogical University, 634041 Tomsk, Russia*

Basic properties of even (odd) supermanifolds endowed with a connection respecting a given symplectic structure are studied. Such supermanifolds can be considered as generalization of Fedosov manifolds to the supersymmetric case.

1 Introduction

The formulation of fundamental physical theories, classical as well as quantum ones, by differential geometric methods nowadays is well established and has a great conceptual virtue. Probably, the most prominent example is the formulation of general relativity on Riemannian manifolds, i.e., the geometrization of gravitational force; no less important is the geometric formulation of gauge field theories of primary interactions on fiber bundles. Another essential route has been opened by the formulation of classical mechanics – and also classical field theories – on symplectic manifolds and their connection with geometric quantization. The properties of such kind of manifolds are widely studied.

Recently, some advanced methods of Lagrangian quantization involve more complicated manifolds namely the so-called Fedosov manifolds, i.e., symplectic manifolds equipped with a symmetric connection which respects the symplectic structure. Fedosov manifolds have been introduced for the first time in the framework of deformation quantization [1]. The properties of Fedosov manifolds have been investigated in detail (see, e.g., Ref. [2]). Especially, let us mention that for any Fedosov manifold the scalar curvature K is trivial, $K = 0$, and that the specific relation $\omega_{ij,kl} = (1/3)R_{klij}$, in terms of normal coordinates, holds between the symplectic structure ω_{ij} and the curvature tensor R_{klij}.

The discovery of supersymmetry [3] enriched modern quantum field theory with the notion of supermanifolds being studied extensively by Berezin [4]. Systematic considerations of supermanifolds and Riemannian supermanifolds were performed by DeWitt [5]. At present, symplectic supermanifolds and the corresponding differential geometry are widely involved and studied in consideration of some problems of modern theoretical and mathematical physics [6, 7].

However, the situation concerning Fedosov supermanifolds is quite different. Only flat even Fedosov supermanifolds have been used in the study of a coordinate-free scheme of deformation quantization [8], for an explicit realization of the extended antibrackets [9] and for the formulation of the modified triplectic quantization in general coordinates, see, [10] and references cited therein. Here, on the basis previous results [11, 12], we give an overview on the present status concerning the structure of arbitrary Fedosov supermanifolds, especially the properties of their curvature tensor and scalar curvature as well as the relations between the supersymplectic structure, the connection and the curvature in normal and general local coordinates.

The paper is organised as follows. In Sect. 2, we give a brief review of the definition of tensor fields on supermanifolds. In Sect. 3, we consider affine connections on a supermanifold and their curvature tensors. In Sect. 4, we present the notion of even (odd) Fedosov supermanifolds and

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[2] E-mail: geyer@itp.uni-leipzig.de

[3] E-mail: lavrov@tspu.edu.ru; lavrov@itp.uni-leipzig.de

of even (odd) symplectic curvature tensors. In Sect. 5, we study the introduction of the Ricci tensor and the property the scalar curvature which is non-trivial for odd Fedosov supermanifolds. In Sect. 6, we introduce normal coordinates on supermanifolds and affine extensions of the Christoffel symbols as well as tensor fields. In Sect. 7, we derive the relation existing between the first order affine extension of the Christoffel symbols and the curvature tensor for any Fedosov supermanifold both in normal coordinates and arbitrary local coordinates. In Sect. 8, we present a relation between the second order affine extension of the symplectic structure and the curvature tensor. In Sect. 9 we give a few concluding remarks.

We use the condensed notation suggested by DeWitt [5]. Derivatives with respect to the coordinates x^i are understood as acting from the left and for them the notation $\partial_i A = \partial A/\partial x^i$ is used. Right derivatives with respect to x^i are labelled by the subscript "r" or the notation $A_{,i} = \partial_r A/\partial x^i$ is used. The Grassmann parity of any quantity A is denoted by $\epsilon(A)$.

2 Tensor fields on supermanifolds

To start with, we review explicitly some of the basic definitions and simple relations of tensor analysis on supermanifolds which are useful in order to avoid elementary pitfalls in the course of the computations. Thereby, we adopt the conventions of DeWitt [5].

Let the variables $x^i, \epsilon(x^i) = \epsilon_i$ be local coordinates of a supermanifold $M, dimM = N$, in the vicinity of a point P. Let the sets $\{e_i := \frac{\partial_r}{\partial x^i}\}$ and $\{e^i := dx^i\}$ be coordinate bases in the tangent space T_PM and the cotangent space T^*_PM, respectively. If one goes over to another set $\bar{x}^i = \bar{x}^i(x)$ of local coordinates the basis vectors in T_PM and T^*_PM transform as follows:

$$\bar{e}_i = e_j \frac{\partial_r x^j}{\partial \bar{x}^i}, \quad \bar{e}^i = e^j \frac{\partial \bar{x}^i}{\partial x^j}. \tag{1}$$

For the transformation matrices the following relations hold:

$$\frac{\partial_r \bar{x}^i}{\partial x^k}\frac{\partial_r x^k}{\partial \bar{x}^j} = \delta^i_j, \quad \frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^k} = \delta^i_j, \quad \frac{\partial_r x^i}{\partial \bar{x}^k}\frac{\partial_r \bar{x}^k}{\partial x^j} = \delta^i_j, \quad \frac{\partial \bar{x}^k}{\partial x^j}\frac{\partial x^i}{\partial \bar{x}^k} = \delta^i_j. \tag{2}$$

Introduce the Cartesian product space Π^n_m

$$\Pi^n_m = \overbrace{T^*_P \times \cdots \times T^*_P}^{n\ times} \times \underbrace{T_P \times \cdots \times T_P}_{m\ times}. \tag{3}$$

Let $\mathbf{T}$ be a mapping $\mathbf{T} : \Pi^n_m \to \Lambda$ that sends every element $(\omega^{i_1}, ..., \omega^{i_n}, X_{j_1}, ..., X_{j_m}) \in \Pi^n_m$ into supernumber $\mathbf{T}(\omega^{i_1}, ..., \omega^{i_n}, X_{j_1}, ..., X_{j_m}) \in \Lambda$ where Λ is the Grassmann algebra. This mapping is said to be a *tensor of rank (n,m) at P* if for all $\omega, \sigma \in T^*_PM$, all $X, Y \in T_PM$ and all $\alpha \in \Lambda$ it satisfies the multilinear laws

$$\begin{aligned}
\mathbf{T}(...\omega + \sigma...) &= \mathbf{T}(...\omega...) + \mathbf{T}(...\sigma...),\\
\mathbf{T}(...X + Y...) &= \mathbf{T}(...X...) + \mathbf{T}(...Y...),\\
\mathbf{T}(...\omega\alpha, \sigma...) &= \mathbf{T}(...\omega, \alpha\sigma...),\\
\mathbf{T}(...\omega\alpha, X...) &= \mathbf{T}(...\omega, \alpha X...),\\
\mathbf{T}(...X\alpha, Y...) &= \mathbf{T}(...X, \alpha Y...),\\
\mathbf{T}(...X\alpha) &= \mathbf{T}(...X)\alpha.
\end{aligned} \tag{4}$$

It is useful to work with components of $\mathbf{T}$ relative to the bases $\{e^i\} and \{e_i\}$

$$T^{i_1...i_n}{}_{j_1...j_m} = \mathbf{T}(e^{i_1}, ..., e^{i_n}, e_{j_1}, ..., e_{j_m}), \quad T_{j_1...j_m}{}^{i_1...i_n} = \mathbf{T}(e_{j_1}, ..., e_{j_m}, e^{i_1}, ..., e^{i_n}). \tag{5}$$

Then a tensor field of type (n, m) with rank $n + m$ is defined as a geometric object which, in each local coordinate system $(x) = (x^1, ..., x^N)$, is given by a set of functions with n upper and m lower indices obeying definite transformation rules. Here we omit the transformation rules for

the components of any tensor under a change of coordinates, $(x) \to (\bar{x})$, referring to [11], and restrict ourselves to the case of the second order tensor only. From (1), (4) and (5) it follows

$$\bar{T}^{ij} = T^{mn}\frac{\partial\bar{x}^j}{\partial x^n}\frac{\partial\bar{x}^i}{\partial x^m}(-1)^{\epsilon_j(\epsilon_i+\epsilon_m)}, \quad \bar{T}_{ij} = T_{mn}\frac{\partial_r x^n}{\partial\bar{x}^j}\frac{\partial_r x^m}{\partial\bar{x}^i}(-1)^{\epsilon_j(\epsilon_i+\epsilon_m)}, \tag{6}$$

$$\bar{T}^i{}_j = T^m{}_n\frac{\partial_r x^n}{\partial\bar{x}^j}\frac{\partial\bar{x}^i}{\partial x^m}(-1)^{\epsilon_j(\epsilon_i+\epsilon_m)}, \quad \bar{T}_i{}^j = T_m{}^n\frac{\partial\bar{x}^j}{\partial x^n}\frac{\partial_r x^m}{\partial\bar{x}^i}(-1)^{\epsilon_j(\epsilon_i+\epsilon_m)}. \tag{7}$$

Note that the unit matrix δ^i_j is connected with unit tensor fields $\delta^i{}_j$ and $\delta_j{}^i$ as follows

$$\delta^i_j = \delta^i{}_j = (-1)^{\epsilon_i}\ \delta_j{}^i = (-1)^{\epsilon_j}\ \delta_j{}^i. \tag{8}$$

From a tensor field of type (n,m) with rank $n+m$, where $n \neq 0,\ m \neq 0$, one can construct a tensor field of type $(n-1,m-1)$ with rank $n+m-2$ by the contraction of an upper and a lower index by the rules

$$T^{i_1\ldots i_{s-1}\,i\,i_{s+1}\ldots i_n}{}_{j_1\ldots j_{q-1}\,i\,j_{q+1}\ldots j_m}\,(-1)^{\epsilon_i(\epsilon_{i_{s+1}}+\cdots+\epsilon_{i_n}+\epsilon_{j_1}+\cdots+\epsilon_{j_{q-1}}+1)}, \tag{9}$$

$$T_{j_1\ldots j_{q-1}\,i\,j_{q+1}\ldots j_m}{}^{i_1\ldots i_{s-1}\,i\,i_{s+1}\ldots i_n}\,(-1)^{\epsilon_i(\epsilon_{i_{s+1}}+\cdots+\epsilon_{i_n}+\epsilon_{j_1}+\cdots+\epsilon_{j_{q-1}})}. \tag{10}$$

In particular, for the tensor fields of type $(1,1)$ the contraction leads to the supertraces,

$$T^i{}_i\,(-1)^{\epsilon_i} \quad \text{and} \quad T_i{}^i. \tag{11}$$

¿From two tensor fields $T^{i_1\cdots i_n}$ and $P_{j_i\ldots j_m}$ of types $(n.0)$ and $(0,m)$ one can construct new tensor fields of type $(n-1,m-1)$ using the multiplication procedure in the following way:

$$(-1)^{\epsilon(P)(\epsilon_{i_1}+\cdots+\epsilon_{i_{n-1}}+\epsilon_k)+\epsilon_k}\ T^{i_1\ldots i_{n-1}k}\ P_{kj_1\ldots j_{m-1}}, \tag{12}$$

$$(-1)^{\epsilon(T)(\epsilon_{j_1}+\cdots+\epsilon_{j_{m-1}}+\epsilon_k)}\ P_{j_1\ldots j_{m-1}k}\ T^{ki_1\ldots i_{n-1}}\ . \tag{13}$$

In particular, for the second rank tensor fields T^{ij} and P_{ij} begineqnarray

$$(-1)^{\epsilon(P)(\epsilon_i+\epsilon_k)+\epsilon_k}T^{ik}P_{kj} \ \text{ or } \ (-1)^{\epsilon(T)(\epsilon_i+\epsilon_k)}P_{ik}T^{kj}. \tag{14}$$

Furthermore, taking into account (14), the unique inverse of a (non-degenerate) second rank tensor field of type (2,0) will be defined as follows:

$$(-1)^{(\epsilon_i+\epsilon_k)\epsilon(T)+\epsilon_k}\ T^{ik}\ (T^{-1})_{kj} = \delta^i{}_j, \quad (-1)^{(\epsilon_j+\epsilon_k)\epsilon(T)}\ (T^{-1})_{jk}\ T^{ki} = \delta_j{}^i\,, \tag{15}$$
$$\epsilon(T^{-1}_{ij}) = \epsilon(T^{ij}) = \epsilon(T) + \epsilon_i + \epsilon_j\,,$$

and correspondingly for tensor fields of type (0,2).

Let us emphasize that the inclusion of the correct sign factors into the definitions of contractions, (9) and (10), and of the inverse tensors, (15), is essential. Namely, let us consider a second rank tensor field of type (2,0) obeying the property of generalized (anti)symmetry,

$$T^{ij}_{\pm} = \pm(-1)^{\epsilon_i\epsilon_j}T^{ji}_{\pm}. \tag{16}$$

Obviously, that property is in agreement with the transformation law (6),

$$\bar{T}^{ij}_{\pm} = T^{mn}_{\pm}\frac{\partial\bar{x}^j}{\partial x^n}\frac{\partial\bar{x}^i}{\partial x^m}(-1)^{\epsilon_j(\epsilon_i+\epsilon_m)} = \pm T^{nm}_{\pm}\frac{\partial\bar{x}^i}{\partial x^m}\frac{\partial\bar{x}^j}{\partial x^n}(-1)^{\epsilon_i\epsilon_n} = \pm(-1)^{\epsilon_i\epsilon_j}\bar{T}^{ji}_{\pm}.$$

Thus, the notion of generalized (anti)symmetry of a tensor field of type (2,0) is invariantly defined in any coordinate system.

Now, suppose that $T^{ij}_{\pm}$ is non-degenerate, thus allowing for the introduction of the corresponding inverse tensor fields of type (0,2) according to (15). From (16) one gets

$$(T^{-1}_{\pm})_{ij} = \pm(-1)^{\epsilon_i\epsilon_j+\epsilon(T)}(T^{-1}_{\pm})_{ji}\,, \tag{17}$$

and, as it should be, also this generalized (anti)symmetry is invariantly defined.

3 Affine connection on supermanifolds and curvature

In analogy to the case of tensor analysis on manifolds, on a supermanifold M one introduces the covariant derivation (or affine connection) as a mapping ∇ (with components ∇_i, $\epsilon(\nabla_i) = \epsilon_i$) from the set of tensor fields on M to itself by the requirement that it should be a tensor operation acting from the right and adding one more lower index and, when it is possible locally to introduce Cartesian coordinates on M, that it should reduce to the usual (right–)differentiation. For arbitrary supermanifolds the covariant derivative ∇ (or connection Γ) is defined through the (right–) differentiation and the separate contraction of upper and lower indices with the connection components accompanig definite numerical factors which depend on the Grassmann parities of local coordinates. More explicitly, they are given as local operations acting on scalar, vector and co-vector fields by the rules

$$T\,\nabla_i = T_{,i}\,, \tag{18}$$

$$T^i\,\nabla_j = T^i{}_{,j} + T^k\Gamma^i{}_{kj}(-1)^{\epsilon_k(\epsilon_i+1)}\,, \tag{19}$$

$$T_i\,\nabla_j = T_{i,j} - T_k\Gamma^k{}_{ij}\,, \tag{20}$$

and on second-rank tensor fields of type $(2,0), (0,2)$ and $(1,1)$ by the rules

$$T^{ij}\,\nabla_k = T^{ij}{}_{,k} + T^{il}\,\Gamma^j{}_{lk}(-1)^{\epsilon_l(\epsilon_j+1)} + T^{lj}\,\Gamma^i{}_{lk}(-1)^{\epsilon_i\epsilon_j+\epsilon_l(\epsilon_i+\epsilon_j+1)}\,, \tag{21}$$

$$T_{ij}\,\nabla_k = T_{ij,k} - T_{il}\,\Gamma^l{}_{jk} - T_{lj}\,\Gamma^l{}_{ik}(-1)^{\epsilon_i\epsilon_j+\epsilon_l\epsilon_j}\,, \tag{22}$$

$$T^i{}_j\,\nabla_k = T^i{}_{j,k} - T^i{}_l\,\Gamma^l{}_{jk} + T^l{}_j\,\Gamma^i{}_{lk}(-1)^{\epsilon_i\epsilon_j+\epsilon_l(\epsilon_i+\epsilon_j+1)}\,. \tag{23}$$

Similarly, the action of the covariant derivative on a tensor field of any rank and type is given in terms of their tensor components, their ordinary derivatives and the connection components.

As usual, the affine connection components do not transform as mixed tensor fields, instead they obtain an additional inhomogeneous term:

$$\bar{\Gamma}^i{}_{jk} = (-1)^{\epsilon_n(\epsilon_m+\epsilon_j)}\frac{\partial_r\bar{x}^i}{\partial x^l}\Gamma^l{}_{mn}\frac{\partial_r x^m}{\partial\bar{x}^j}\frac{\partial_r x^n}{\partial\bar{x}^k} + \frac{\partial_r\bar{x}^i}{\partial x^m}\frac{\partial_r^2 x^m}{\partial\bar{x}^j\partial\bar{x}^k}\,. \tag{24}$$

In general, the connection components $\Gamma^i{}_{jk}$ do not have the property of (generalized) symmetry w.r.t. the lower indices. The deviation from this symmetry is the torsion,

$$T^i{}_{jk} := \Gamma^i{}_{jk} - (-1)^{\epsilon_j\epsilon_k}\Gamma^i{}_{kj}\,, \tag{25}$$

which transforms as a tensor field. If the supermanifold M is torsionless, i.e., $T^i{}_{jk} = 0$, then one says that a symmetric connection is defined on M. Here, with the aim of studying Fedosov supermanifolds, we consider only symmetric connections.

The Riemannian tensor field $R^i{}_{mjk}$, according to Ref. [5], is defined in a coordinate basis by the action of the commutator of covariant derivatives, $[\nabla_i, \nabla_j] = \nabla_i\nabla_j - (-1)^{\epsilon_i\epsilon_j}\nabla_j\nabla_i$, on a vector field T^i as follows:

$$T^i[\nabla_j, \nabla_k] = -(-1)^{\epsilon_m(\epsilon_i+1)}T^m R^i{}_{mjk}. \tag{26}$$

A straightforward calculation yields

$$R^i{}_{mjk} = -\Gamma^i{}_{mj,k} + \Gamma^i{}_{mk,j}(-1)^{\epsilon_j\epsilon_k} + \Gamma^i{}_{jn}\Gamma^n{}_{mk}(-1)^{\epsilon_j\epsilon_m} - \Gamma^i{}_{kn}\Gamma^n{}_{mj}(-1)^{\epsilon_k(\epsilon_m+\epsilon_j)}. \tag{27}$$

The Riemannian tensor field possesses the following generalized antisymmetry property,

$$R^i{}_{mjk} = -(-1)^{\epsilon_j\epsilon_k}R^i{}_{mkj}\,; \tag{28}$$

furthermore, it obeys the (super) Jacobi identity,

$$(-1)^{\epsilon_m\epsilon_k}R^i{}_{mjk} + (-1)^{\epsilon_j\epsilon_m}R^i{}_{jkm} + (-1)^{\epsilon_k\epsilon_j}R^i{}_{kmj} \equiv 0 \tag{29}$$

and the (super) Bianchi identity,

$$(-1)^{\epsilon_i\epsilon_j}R^n{}_{mjk;i} + (-1)^{\epsilon_i\epsilon_k}R^n{}_{mij;k} + (-1)^{\epsilon_k\epsilon_j}R^n{}_{mki;j} \equiv 0\,, \tag{30}$$

with the notation $R^n{}_{mjk;i} := R^n{}_{mjk}\nabla_i$.

4 Fedosov supermanifolds

Suppose now we are given an even (odd) symplectic supermanifold, (M,ω) with an even (odd) symplectic structure ω, $\epsilon(\omega)=0$ (or 1). Let ∇ (or Γ) be a covariant derivative (connection) on M which preserves the 2-form ω, $\omega\nabla=0$. In a coordinate basis this requirement reads

$$\omega_{ij,k}-\omega_{im}\Gamma^m{}_{jk}+\omega_{jm}\Gamma^m{}_{ik}(-1)^{\epsilon_i\epsilon_j}=0. \tag{31}$$

If, in addition, Γ is symmetric then we have an even (odd) symplectic connection (or symplectic covariant derivative) on M. Now, a *Fedosov supermanifold* (M,ω,Γ) is defined as a symplectic supermanifold with a given symplectic connection.

Let us introduce the curvature tensor of a symplectic connection with all indices lowered,

$$R_{ijkl}=\omega_{in}R^n{}_{jkl},\quad \epsilon(R_{ijkl})=\epsilon(\omega)+\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_l, \tag{32}$$

where $R^n{}_{jkl}$ is given by (27). This leads to the following representation,

$$R_{imjk}=-\omega_{in}\Gamma^n{}_{mj,k}+\omega_{in}\Gamma^n{}_{mk,j}(-1)^{\epsilon_j\epsilon_k}+\Gamma_{ijn}\Gamma^n{}_{mk}(-1)^{\epsilon_j\epsilon_m}-\Gamma_{ikn}\Gamma^n{}_{mj}(-1)^{\epsilon_k(\epsilon_m+\epsilon_j)}\,, \tag{33}$$

where we used the notation

$$\Gamma_{ijk}=\omega_{in}\Gamma^n{}_{jk},\quad \epsilon(\Gamma_{ijk})=\epsilon(\omega)+\epsilon_i+\epsilon_j+\epsilon_k\,. \tag{34}$$

Using this, the relation (31) reads $\omega_{ij,k}=\Gamma_{ijk}-\Gamma_{jik}(-1)^{\epsilon_i\epsilon_j}$. Furthermore, from Eq. (33) it is obvious that

$$R_{ijkl}=-(-1)^{\epsilon_k\epsilon_l}R_{ijlk}, \tag{35}$$

and, using (32) and (29), one deduces the (super) Jacobi identity for R_{ijkl},

$$(-1)^{\epsilon_j\epsilon_l}R_{ijkl}+(-1)^{\epsilon_l\epsilon_k}R_{iljk}+(-1)^{\epsilon_k\epsilon_j}R_{iklj}=0\,. \tag{36}$$

In addition, the curvature tensor R_{ijkl} is (generalized) symmetric w.r.t. the first two indices (see [11]),

$$R_{ijkl}=(-1)^{\epsilon_i\epsilon_j}R_{jilk}. \tag{37}$$

For any even (odd) symplectic connection there holds the identity

$$(-1)^{\epsilon_i\epsilon_l}R_{ijkl}+(-1)^{\epsilon_l\epsilon_k+\epsilon_l\epsilon_j}R_{lijk}+(-1)^{\epsilon_k\epsilon_j+\epsilon_l\epsilon_j+\epsilon_i\epsilon_k}R_{klij}+(-1)^{\epsilon_i\epsilon_j+\epsilon_i\epsilon_k}R_{jkli}=0. \tag{38}$$

This is proved by using the Jacobi identity (36) together with a cyclic change of the indices. In the identity (38) the components of the symplectic curvature tensor occur with cyclic permutations of all the indices (on R). However, the pre-factors depending on the Grassmann parities of indices are not obtained by cyclic permutation.

5 Ricci and scalar curvature tensors

Having the curvature tensor, R_{ijkl}, and the tensor field ω^{ij}, which is inverse to ω_{ij},

$$\omega^{ik}\,\omega_{kj}(-1)^{\epsilon_k+\epsilon(\omega)(\epsilon_i+\epsilon_k)}=\delta^i_j,\quad (-1)^{\epsilon_i+\epsilon(\omega)(\epsilon_i+\epsilon_k)}\omega_{ik}\,\omega^{kj}=\delta^j_i, \tag{39}$$

$$\omega^{ij}=-(-1)^{\epsilon_i\epsilon_j+\epsilon(\omega)}\omega^{ji},\quad \epsilon(\omega^{ij})=\epsilon(\omega)+\epsilon_i+\epsilon_j, \tag{40}$$

one can define the following three different tensor fields of type $(0,2)$,

$$R_{ij}=\omega^{kn}R_{nkij}(-1)^{(\epsilon(\omega)+1)(\epsilon_k+\epsilon_n)}\quad=\ R^k{}_{kij}\,(-1)^{\epsilon_k}\,, \tag{41}$$

$$K_{ij}=\omega^{kn}R_{nikj}(-1)^{\epsilon_i\epsilon_k+(\epsilon(\omega)+1)(\epsilon_k+\epsilon_n)}\ =\ R^k{}_{ikj}\,(-1)^{\epsilon_k(\epsilon_i+1)}\,, \tag{42}$$

$$Q_{ij}=\omega^{kn}R_{ijnk}(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_n)+(\epsilon(\omega)+1)(\epsilon_k+\epsilon_n)}\,, \tag{43}$$

$$\epsilon(R_{ij})=\epsilon(K_{ij})=\epsilon(Q_{ij})=\epsilon_i+\epsilon_j\,.$$

From the definitions (41), (43) and the symmetry properties of R_{ijkl}, it follows immediately that for any symplectic connection one has $R_{ij} = -(-1)^{\epsilon_i\epsilon_j}R_{ji}$ and $Q_{ij} = (-1)^{\epsilon_i\epsilon_j}Q_{ji}$. Moreover we obtain the relations

$$[1 + (-1)^{\epsilon(\omega)}]R_{ij} = 0, \quad [1 - (-1)^{\epsilon(\omega)}]Q_{ij} = 0. \tag{44}$$

¿From (38) and (41) – (43) it follows the relations

$$R_{ij} + Q_{ij} + (-1)^{\epsilon_i\epsilon_j}K_{ji} + (-1)^{\epsilon(\omega)}K_{ij} = 0, \tag{45}$$

$$[1 + (-1)^{\epsilon(\omega)}]\,(K_{ij} - (-1)^{\epsilon_i\epsilon_j}K_{ji}) = 0. \tag{46}$$

Therefore for any even symplectic connection we obtain

$$K_{ij} = (-1)^{\epsilon_i\epsilon_j}K_{ji}, \quad R_{ij} = 0, \quad Q_{ij} = -2K_{ij}, \tag{47}$$

while for any odd symplectic connection we have

$$Q_{ij} = 0, \quad R_{ij} = K_{ij} - (-1)^{\epsilon_i\epsilon_j}K_{ji}. \tag{48}$$

The tensor field K_{ij} should be considered as the only independent second-rank tensor which can be constructed from the symplectic curvature. We refer to K_{ij} as the *Ricci tensor* of an even (odd) Fedosov supermanifold. Notice that in the odd case K_{ij} has no special symmetry property.

Let us define the *scalar curvature* K of a Fedosov supermanifold by the formula

$$K = \omega^{ji}K_{ij}(-1)^{\epsilon_i+\epsilon_j} = \omega^{ji}\omega^{kn}R_{nikj}(-1)^{\epsilon_i+\epsilon_j+\epsilon_i\epsilon_k+(\epsilon_k+\epsilon_n)(\epsilon(\omega)+1)}. \tag{49}$$

From the symmetry properties of R_{ijkl} and ω^{ij}, it follows that on any Fedosov supermanifold one has

$$[1 + (-1)^{\epsilon(\omega)}]K = 0. \tag{50}$$

Therefore, as is the case for ordinary Fedosov manifolds [2], for any even symplectic connection the scalar curvature necessarily vanishes. But the situation becomes different for odd Fedosov supermanifold where no restriction on the scalar curvature occurs. Therefore, in contrast to both the usual Fedosov manifolds and the even Fedosov supermanifolds, any odd Fedosov supermanifolds can be characterized by the scalar curvature as an additional geometrical structure [11]. This basic property of the scalar curvature can be used to formulate the following statement [13]: In both, the even and odd cases there exists the relation $K^2 = 0$, and therefore any regular function of the scalar curvature on any Fedosov supermanifolds belongs to class of linear functions $\phi(x) = f[K(x)] = \alpha + \beta\,K(x)$.

6 Affine extensions of Christoffel symbols and tensors on symplectic supermanifolds

In Ref. [2] the virtues of using normal coordinates for studying the properties of Fedosov manifolds was demonstrated. Here, following Ref. [12], we are going to extend this method on Fedosov supermanifolds (M, ω, Γ) [12]. Normal coordinates $\{y^i\}$ within a point $p \in M$ can be introduced by using the geodesic equations as those local coordinates which satisfy the relations (p corresponds to $y = 0$)

$$\Gamma^i{}_{jk}(y)\,y^k\,y^j = 0, \quad \epsilon(\Gamma_{ijk}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k. \tag{51}$$

It follows from (51) and the symmetry properties of Γ_{ijk} w.r.t. $(j\,k)$ that

$$\Gamma_{ijk}(0) = 0. \tag{52}$$

In normal coordinates there exist additional relations at p containing the partial derivatives of Γ_{ijk}. Namely, consider the Taylor expansion of $\Gamma_{ijk}(y)$ at $y=0$,

$$\Gamma_{ijk}(y)=\sum_{n=1}^{\infty}\frac{1}{n!}A_{ijkj_1\ldots j_n}y^{j_n}\cdots y^{j_1},\quad \text{where}\quad A_{ijkj_1\ldots j_n}=A_{ijkj_1\ldots j_n}(p)=\left.\frac{\partial_r^n\Gamma_{ijk}}{\partial y^{j_1}\ldots\partial y^{j_n}}\right|_{y=0} \tag{53}$$

is called an *affine extension* of Γ_{ijk} of order $n=1,2,\ldots$. The symmetry properties of $A_{ijkj_1\ldots j_n}$ are evident from their definition (53), namely, they are (generalized) symmetric w.r.t. $(j\,k)$ as well as $(j_1\ldots j_n)$. The set of all affine extensions of Γ_{ijk} uniquely defines a symmetric connection according to (53) and satisfy an infinite sequence of identities [12]. In the lowest nontrivial order they have the form

$$A_{ijkl}+A_{ijlk}(-1)^{\epsilon_k\epsilon_l}+A_{iklj}(-1)^{\epsilon_j(\epsilon_l+\epsilon_k)}=0. \tag{54}$$

Analogously, the affine extensions of an arbitrary tensor $T=(T^{i_1\ldots i_k}{}_{m_1\ldots m_l})$ on M are defined as tensors on M whose components at $p\in M$ in the local coordinates $(x^1,\ldots,x^{2N})$ are given by the formula

$$T^{i_1\ldots i_k}{}_{m_1\ldots m_l,j_1\ldots j_n}\equiv T^{i_1\ldots i_k}{}_{m_1\ldots m_l,j_1\ldots j_n}(0)=\left.\frac{\partial_r^n T^{i_1\ldots i_k}{}_{m_1\ldots m_l}}{\partial y^{j_1}\ldots\partial y^{j_n}}\right|_{y=0} \tag{55}$$

where $(y^1,\ldots,y^{2N})$ are normal coordinates associated with $(x^1,\ldots,x^{2N})$ at p. The first extension of any tensor coincides with its covariant derivative because $\Gamma^i{}_{jk}(0)=0$ in normal coordinates.

In the following, any relation containing affine extensions are to be understood as holding in a neighborhood U of an arbitrary point $p\in M$. Let us also observe the convention that, if a relation holds for arbitrary local coordinates, the arguments of the related quantities will be suppressed.

7 First order affine extension of Christoffel symbols and curvature tensor of Fedosov supermanifolds

For a given Fedosov supermanifold (M,ω,Γ), the symmetric connection Γ respects the symplectic structure ω [10]:

$$\omega_{ij,k}=\Gamma_{ijk}-\Gamma_{jik}(-1)^{\epsilon_i\epsilon_j}. \tag{56}$$

Therefore, among the affine extensions of ω_{ij} and Γ_{ijk} there must exist some relations. Introducing the affine extensions of ω_{ij} in the normal coordinates $(y^1,\ldots,y^{2N})$ at $p\in M$ according to,

$$\omega_{ij}(y)=\sum_{n=1}^{\infty}\frac{1}{n!}\Omega_{ij,j_1\ldots j_n}\,y^{j_n}\cdots y^{j_1},\quad \Omega_{ij,j_1\ldots j_n}=\omega_{ij,j_1\ldots j_n}(0). \tag{57}$$

Using the symmetry properties of $\omega_{ij,j_1\ldots j_n}(0)$ one easily obtains the Taylor expansion for $\omega_{ij,k}$:

$$\omega_{ij,k}(y)=\sum_{n=1}^{\infty}\frac{1}{n!}\Omega_{ij,kj_1\ldots j_n}\,y^{j_n}\cdots y^{j_1}. \tag{58}$$

Taking into account (56) and comparing (53) and (58) we obtain

$$\Omega_{ij,kj_1\ldots j_n}=A_{ijkj_1\ldots j_n}-A_{jikj_1\ldots j_n}(-1)^{\epsilon_i\epsilon_j}; \tag{59}$$

in particular,

$$\Omega_{ij,kl}=A_{ijkl}-A_{jikl}(-1)^{\epsilon_i\epsilon_j}. \tag{60}$$

Now, consider the curvature tensor R_{ijkl} in the normal coordinates at $p \in M$. Then, due to $\Gamma_{ijk}(p) = 0$, we obtain the following representation of the curvature tensor in terms of the affine extensions of the symplectic connection

$$R_{ijkl}(0) = -A_{ijkl} + A_{ijlk}(-1)^{\epsilon_k \epsilon_l}. \tag{61}$$

Taking into account (54) and (61) a relation containing the curvature tensor and the first affine extension of Γ can be derived Indeed, the desired relation obtains as follows

$$A_{ijkl} \equiv \Gamma_{ijk,l}(0) = -\frac{1}{3}\left[R_{ijkl}(0) + R_{ikjl}(0)(-1)^{\epsilon_k \epsilon_j}\right], \tag{62}$$

where the antisymmetry (35) of the curvature tensor were used.

Notice, that relation (62) was derived in normal coordinates. It seems to be of general interest to find its analog relation in terms arbitrary local coordinates (x) because the Christoffel symbols are not tensors while the r.h.s. of (62) is a tensor. Under that change of coordinates $(x) \to (y)$ in some vicinity U of p the Christoffel symbols Γ_{ijk} transform according to the rule

$$\Gamma_{ijk}(y) = \left(\Gamma_{pqr}(x)\frac{\partial_r x^r}{\partial y^k}\frac{\partial_r x^q}{\partial y^j}(-1)^{\epsilon_k(\epsilon_j+\epsilon_q)} + \ \omega_{pq}(x)\frac{\partial_r^2 x^q}{\partial y^j \partial y^k}\right)\frac{\partial_r x^p}{\partial y^i}(-1)^{(\epsilon_k+\epsilon_j)(\epsilon_i+\epsilon_p)}. \tag{63}$$

In its turn the matrix of second derivatives can be expressed in the form

$$\frac{\partial_r^2 x^q}{\partial y^j \partial y^k} = \frac{\partial_r x^q}{\partial y^l}\Gamma^l{}_{jk}(y) - \Gamma^q{}_{lm}(x)\frac{\partial_r x^m}{\partial y^k}\frac{\partial_r x^l}{\partial y^j}(-1)^{\epsilon_k(\epsilon_j+\epsilon_l)}. \tag{64}$$

In particular at $p \in M$ $(y=0)$ we have the relation

$$\left(\frac{\partial_r^2 x^q}{\partial y^j \partial y^k}\right)_0 = -\Gamma^q{}_{lm}(x_0)\left(\frac{\partial_r x^m}{\partial y^k}\right)_0\left(\frac{\partial_r x^l}{\partial y^j}\right)_0(-1)^{\epsilon_k(\epsilon_j+\epsilon_l)} \equiv -\Gamma^p{}_{jk}(x_0). \tag{65}$$

Differentiating (63) with respect to y we find

$$\begin{aligned}
\Gamma_{ijk,l}(y) = \ & \Gamma_{pqr;s}(x)\frac{\partial_r x^s}{\partial y^l}\frac{\partial_r x^r}{\partial y^k}\frac{\partial_r x^q}{\partial y^j}\frac{\partial_r x^p}{\partial y^i}(-1)^{(\epsilon_j+\epsilon_k+\epsilon_l)(\epsilon_i+\epsilon_p)+(\epsilon_k+\epsilon_l)(\epsilon_j+\epsilon_q)+\epsilon_l(\epsilon_k+\epsilon_r)} \\
& + \omega_{pq,r}(x)\frac{\partial_r x^r}{\partial y^l}\frac{\partial_r^2 x^q}{\partial y^j \partial y^k}\frac{\partial_r x^p}{\partial y^i}(-1)^{(\epsilon_k+\epsilon_j)(\epsilon_i+\epsilon_p)+\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_q+\epsilon_p)} \\
& + \omega_{pq}(x)\frac{\partial_r^2 x^q}{\partial y^j \partial y^k}\frac{\partial_r^2 x^p}{\partial y^i \partial y^l}(-1)^{(\epsilon_k+\epsilon_j)(\epsilon_i+\epsilon_p)} + \omega_{pq}(x)\frac{\partial_r^3 x^q}{\partial y^j \partial y^k \partial y^l}\frac{\partial_r x^p}{\partial y^i}(-1)^{(\epsilon_j+\epsilon_k+\epsilon_l)(\epsilon_i+\epsilon_p)},
\end{aligned}$$

where the covariant derivative (for arbitrary local coordinates) is defined by

$$\Gamma_{pqr;s} = \Gamma_{pqr,s} - \Gamma_{pqn}\Gamma^n{}_{rs} - \Gamma_{pnr}\Gamma^n{}_{qs}(-1)^{\epsilon_r(\epsilon_n+\epsilon_q)} - \Gamma_{nqr}\Gamma^n{}_{ps}(-1)^{(\epsilon_r+\epsilon_q)(\epsilon_n+\epsilon_p)}. \tag{66}$$

Restricting to the point $p \in M$ we get

$$\Gamma_{ijk,l}(0) = \left(\Gamma_{ijk;l}(x_0) - \Gamma_{iln}(x_0)\Gamma^n{}_{jk}(x_0)(-1)^{\epsilon_l(\epsilon_j+\epsilon_k)}\right) + \omega_{iq}(x_0)\left(\frac{\partial_r^3 x^q}{\partial y^j \partial y^k \partial y^l}\right)_0. \tag{67}$$

Due to (67) and the identity (54), the matrix of third derivatives at p obeys the following relation,

$$\begin{aligned}
\omega_{iq}(x_0)\left(\frac{\partial_r^3 x^q}{\partial y^j \partial y^k \partial y^l}\right)_0 = -\frac{1}{3}\Big[& \left(\Gamma_{ijk;l} - \Gamma_{ijn}\Gamma^n{}_{kl}\right)(-1)^{\epsilon_j \epsilon_l} + \left(\Gamma_{ilj;k} - \Gamma_{iln}\Gamma^n{}_{jk}\right)(-1)^{\epsilon_l \epsilon_k} \\
& + \left(\Gamma_{ikl;j} - \Gamma_{ikn}\Gamma^n{}_{lj}\right)(-1)^{\epsilon_k \epsilon_j}\Big](x_0)\ (-1)^{\epsilon_j \epsilon_l}.
\end{aligned} \tag{68}$$

With the help of (68) we get the following transformation law for $\Gamma_{ijk;l}$ under change of coordinates at the point p

$$\Gamma_{ijk,l}(0) = \Big[\Gamma_{ijk,l}(x_0) - \frac{1}{3}Z_{ijkl}(x_0)\Big], \tag{69}$$

with the abbreviation

$$Z_{ijkl} = \Gamma_{ijk;l} + \Gamma_{ijl;k}(-1)^{\epsilon_k\epsilon_l} + \Gamma_{ikl;j}(-1)^{(\epsilon_k+\epsilon_l)\epsilon_j} \\ +2\Gamma_{iln}\Gamma^n{}_{jk}(-1)^{(\epsilon_k+\epsilon_j)\epsilon_l} - \Gamma_{ikn}\Gamma^n{}_{jl}(-1)^{\epsilon_j\epsilon_k} - \Gamma_{ijn}\Gamma^n{}_{kl} \tag{70}$$

In straightforward manner one can check that the relations (67) reproduce the correct transformation law for the curvature tensor R_{ijkl}. Therefore, relation (62) is to be generalized as

$$\Gamma_{ijk;l} - \frac{1}{3}Z_{ijkl} = -\frac{1}{3}[R_{ijkl} + R_{ikjl}(-1)^{\epsilon_j\epsilon_k}]. \tag{71}$$

The last equation gets an identity when using the definition (70) of $Z_{ijkl}(x)$ and relation for $R_{ijkl}(x)$ on the r.h.s..

8 Second and third order affine extension of symplectic structure and curvature tensor on Fedosov supermanifolds

Now, let us consider the relation between the second order affine extension of symplectic structure and the symplectic curvature tensor. It is easily found by taking into account (60) and (62). Indeed, the Jacobi identity (36), we obtain

$$\omega_{ij,kl}(0) = A_{ijkl} - A_{jikl}(-1)^{\epsilon_i\epsilon_j} = \frac{1}{3}R_{klij}(0)(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)},$$

Again, since $p \in M$ is arbitrary, we finally obtain its generalization for any local coordinates x:

$$\omega_{ij,k;l} - \frac{1}{3}\left[Z_{ijkl} - Z_{jikl}(-1)^{\epsilon_i\epsilon_j}\right] = \frac{1}{3}(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}R_{klij}. \tag{72}$$

Furthermore, using the second Bianchi identity [11] one gets a relation between the first derivative of the curvature tensor and the affine connections,

$$R_{ijkl,m} = -A_{ijklm} + A_{ijlkm}(-1)^{\epsilon_l\epsilon_k}. \tag{73}$$

as well as the third affine extension of the symplectic structure

$$\omega_{ij,klm} = -\frac{1}{6}[R_{ikjl,m}(-1)^{\epsilon_j\epsilon_k} + R_{ikjm,l}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,k}(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_l+\epsilon_m)} - \\ - R_{jkim,l}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)} - R_{jkim,l}(-1)^{\epsilon_m\epsilon_l+\epsilon_i(\epsilon_j+\epsilon_k)} - R_{jlim,k}(-1)^{\epsilon_k(\epsilon_m+\epsilon_l)+\epsilon_i(\epsilon_j+\epsilon_l)}].$$

In local coordinates (x) the following identity can be proven:

$$R_{mjik;l}(-1)^{\epsilon_j(\epsilon_i+\epsilon_k)} - R_{mijl;k}(-1)^{\epsilon_k(\epsilon_l+\epsilon_j)} + R_{mkjl;i}(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)} - R_{mlik;j}(-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k)} = 0.$$

For the derivation of these relations, see, Ref. [12].

9 Summary

We have considered some properties of tensor fields defined on supermanifolds M. It was shown that only the generalized (anti)symmetry of tensor fields has an invariant meaning, and that differential geometry on supermanifolds should be constructed in terms of such tensor fields.

Any supermanifold M can be equipped with a symmetric connection Γ (covariant derivative ∇). The Riemannian tensor $R^i{}_{jkl}$ corresponding to this symmetric connection Γ satisfies both the (super) Jacobi identity and the (super) Bianchi identity.

Any even (odd) symplectic supermanifold can be equipped with a symmetric connection respecting the given symplectic structure. Such a symmetric connection is called a symplectic connection. The triplet (M, ω, Γ) is called an even (odd) Fedosov supermanifold. The curvature tensor R_{ijkl} of a symplectic connection obeys the property of generalized symmetry with respect

to the first two indices, and the property of generalized antisymmetry with respect to the last two indices. The tensor R_{ijkl} satisfies the Jacobi identity and the specific (for the symplectic geometry) identity (see (38)) containing the sum of components of this tensor with a cyclic permutation of all the indices, which, however, does not (!) contain cyclic permuted factors depending on the Grassmann parities of the indices.

On any even (odd) Fedosov manifold, the Ricci tensor K_{ij} can be defined. In the even case, the Ricci tensor obeys the property of generalized symmetry and gives a trivial result for the scalar curvature. On the contrary, in the odd case the scalar curvature, in general, is nontrivial.

Using normal coordinates on a supermanifold equipped with a symmetric connection we have found relations among the first order affine extensions of the Christoffel symbols and the curvature tensor, the second order affine extension of symplectic structure and the curvature tensor. In similar way it is possible to find relations containing higher order affine extensions of sypmlectic structure, the Christoffel symbols and the curvature tensor. We have established the form of the obtained relations in any local coordinates (see (71), (72)). It was shown that $\Gamma_{ijk;l}(x) - 1/3 Z_{ijkl}(x)$ is a tensor field in terms of which the relations obtained for general local coordinates can be presented, cf. Eq. (69).

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